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FINITE ELEMENT MASS MATRIX LUMPING BY NUMERICAL INTEGRATION WITH NO CONVERGENCE RATE LOSS

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Abstract—Using numerical integration in the formation of the finite element mass matrix and placing the movable nodes at integration points causes it to become lumped or diagonal (block diagonal) with the optimal rate of energy convergence retained.

LUMPING

Application of the standard finite element method to eigenproblems produces the algebraic $Kx = \lambda Mx$ with a (consistent) non-diagonal matrix M. The desire to alter the method to deliver a diagonal (lumped) M is understandable in view of the computational simplifications and savings that ensue from that.

Numerical integration is a technically convenient, routinely used device in the finite element method and can expediently be used[1] to lump the mass matrix with no accuracy loss.

Both in statics and dynamics in order to maintain with numerical integration[2–5] the full rate of energy convergence of which the element is capable all the energy terms need be integrated by a scheme of order $2(p-m)$, p being the degree of the complete polynomial in the shape functions and m the highest differentiation order in the energy expression. Since the mass matrix originates from the integration of only the displacements squared and not their derivatives, placing the element nodes at integration points will result in diagonal element and global mass matrices with the nodal masses being the integration weights. If the weights and nodal point locations can be adjusted to provide a numerical integration scheme of order $2(p-m)$ the optimal rate of convergence is anticipated. The integration scheme for the mass matrix need not be used to integrate the potential energy in the formation of the element stiffness matrix and a more efficient scheme could be employed. But once the location of the nodes have been decided upon all element matrices have to be formed relative to them.

To fix ideas consider the cubic ($p = 3$) string ($m = 1$), four nodal point element in Fig. 1. To retain the sixth order accuracy[6, 7] of this element a fourth order numerical integration scheme is called for. The two end nodes are fixed and only the masses can be varied at them while the interior nodes can be displaced and the masses at them varied. Presently the integration scheme becomes

$$\int_{-1}^{1} u(x)\,dx = m_1[u(-1)+u(1)]+m_2[u(-c)+u(c)] \tag{1}$$

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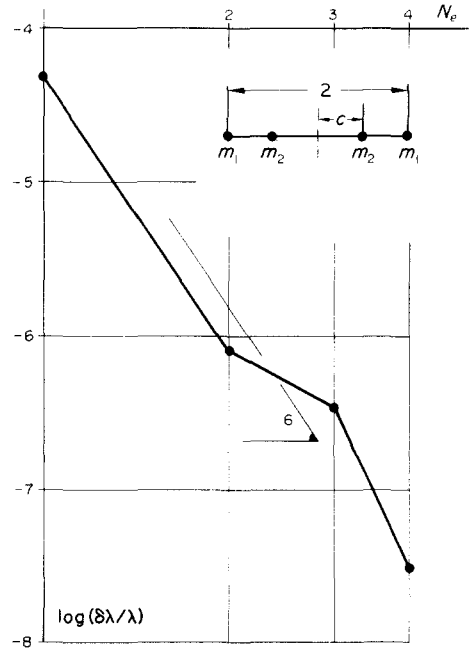

Fig. 1. Convergence of the fundamental eigenvalue λ of a fixed string discretized with the shown element vs the number of elements N_e per half span.

and m_1, m_2 and c are determined by the condition that the integration scheme in equation (1) be exact for a polynomial u of degree 5, yielding $c = \pm\sqrt{(5/5)}$, $m_1 = 1/6$ and $m_2 = 5/6$, a scheme attributed to Lobatto.

Figure 1 describes the convergence of the first eigenvalue ($\lambda = \pi^2$) of a fixed string discretized with this matrix vs the number of elements N_e, and indeed the optimal rate of convergence $O(N_e^{-6})$ is practically achieved.

In higher dimensions some of the nodal masses may vanish or even become negative as in the membrane elements of Fig. 2, causing the higher, uninteresting, portion of the spectrum to become either infinite or negative. (Straight and curved multi-dimensional elements in which the nodal points pattern permits a product integration rule will have positive masses; the product of

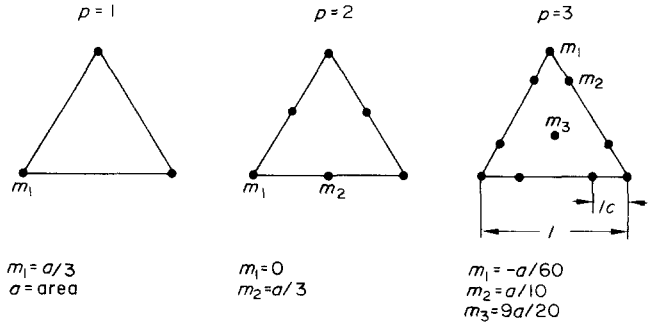

Fig. 2. Triangular membrane elements of the first ($p = 1$), second ($p = 2$) and third order. The mass m_3 is at the center of area and $c = \sqrt{3}(\sqrt{3} - 1)/6$.

the corresponding one-dimensional ones.) To see the effect of zero and negative masses on the spectrum the problem of a square membrane, fixed and free, was solved with the quadratic ($p = 2$) and cubic ($p = 3$) elements of Fig. 2.

Figure 3 describes the convergence of the first eigenvalue λ (exact $2\pi^2$) with the number of elements per side N_{es}. Curve (a) refers to discretization with a consistent mass matrix and (b) to the lumped. In both cases the optimal rate $O(N_{es}^{-4})$ is achieved, with the lumped yielding a somewhat better approximation than the consistent. The consistent mass matrix results from the strict application of the finite element (Rayleigh–Ritz) method and provides therefore an upper bound on the exact eigenvalues. This bound is lost with the lumped formulation and the approximate eigenvalues may decline and yield a better accuracy.

The behavior of the higher eigenvalues can be observed in Table 1 in which the 11 first eigenvalue of the fixed membrane with quadratic elements and a 6×6 mesh is given for both the consistent and lumped formulations. The consistent formulation furnishes in this case 36 eigenvalues and the lumped 27 with the rest being infinite.

Figure 4 and Table 2 give similar results for the cubic element.

Table 3 list all the eigenvalues sorted according to magnitude, of a free–free membrane discretized by a 1×1 mesh of cubic triangular elements to show the distribution of negative eigenvalues.

The influence of lumping on the accuracy of the eigenvectors is to be seen in Tables 4 and 5.

To compute all eigenvalues of $Kx = \lambda Mx$ for the present experimental purposes, the general eigenvalue problem could have been transformed into that of finding the eigenvalues of either $L^{-1}ML^{-T}$ or $L^{T}M^{-1}L$ in which the triangular matrix L is such that $LL^{T} = K$, and solved with

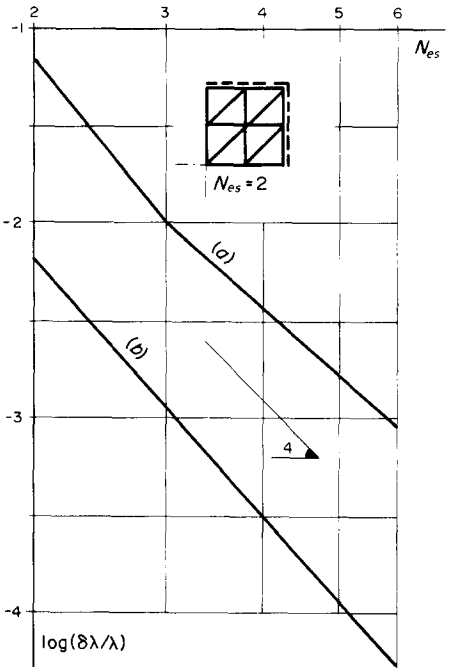

Fig. 3. Convergence of λ in a fixed square membrane discretized with quadratic triangular elements vs the number of elements per side N_{es}. Curves (a) and (b) refer to a consistent and lumped formulation, respectively.

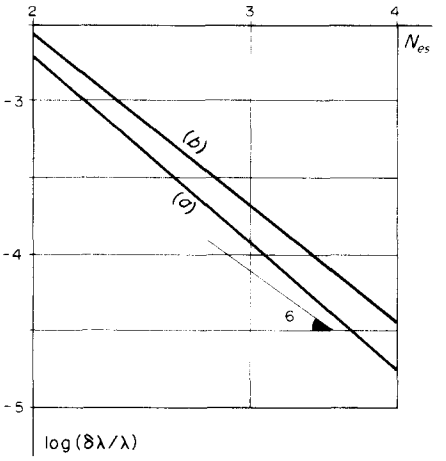

Fig. 4. Same as Fig. 3 but now with cubic elements.

Table 1. Eigenvalues of a fixed membrane discretized by quadratic ($p = 2$) triangular membrane elements. Values are for a 6×6 mesh

Mode	Exact	Lumped	Consistent
1,1	19·739	19·738	19·758
3,1	98·70	98·22	99·74
1,3	98·70	98·22	100·07
3,3	177·7	176·4	185·3
5,1	256·6	242·7	272·8
1,5	256·6	242·7	273·0
3,5	335·6	318·7	365·4
5,3	335·6	318·7	389·8
5,5	493·5	444·6	565·2
7,1	493·5	444·6	570·0
1,7	493·5	451·0	604·1

Table 2. Eigenvalues of a fixed membrane discretized by cubic ($p = 3$) triangular membrane elements. Values are for a 4×4 mesh

Mode	Exact	Lumped	Consistent
1,1	19·7392	19·7399	19·7396
3,1	98·70	98·77	98·80
1,3	98·70	98·84	98·87
3,3	177·65	179·92	179·50
5,1	256·6	250·9	264·4
1,5	256·6	255·3	264·7
3,5	335·6	317·9	348·3
5,3	335·6	347·4	382·0
5,5	493·5	450·1	544·3
7,1	493·5	456·4	553·3
1,7	495·3	517·5	553·3

Table 3. All 15 eigenvalues of a free–free membrane discretized with a 1×1 mesh of triangular cubic elements

Mode	Exact	Lumped	Consistent
0,1	9·87	9·67	9·88
1,0	9·87	9·77	9·88
1,1	19·7	18·9	21·3
2,0	39·5	49·5	48·8
0,2	39·5	50·0	54·9
1,2	49·3	57·6	61·8
2,1	49·3	60·0	66·5
2,2	79	60	128
0,3	89	65	170
3,0	89	−66	170
3,1	99	−78	178
1,3	99	83	190
2,3	128	123	249
3,2	128	−149	353
0,4	158	−149	400

Table 4. l_2 and angular (degrees) error in l_2-normalized eigenvectors in a fixed membrane discretized by cubic ($p = 3$) triangular elements. Values are for a 2×2 mesh using symmetry

Mode	Exact eigenvalue	Error norm		Angular error	
		Lumped	Consistent	Lumped	Consistent
1,1	19·739	0·006	0·003	0·37	0·16
3,3	177·65	0·22	0·16	24	17

Table 5. Same as Table 4 but for a 4×4 mesh

Mode	Exact eigenvalue	Error norm		Angular error	
		Lumped	Consistent	Lumped	Consistent
1,1	19·739	0·0002	0·0003	0·02	0·03
3,3	177·65	0·02	0·01	2·3	2·0

any method applicable to real symmetric matrices. In free–free vibration problems where both M and K might be singular inverse iteration could be applied to the shifted problem $(K + \lambda_0 M)x = \mu Mx$, λ_0 being the shift.

In practice the entire spectrum is not required and both inverse iteration and residual and gradient methods, which are often used to extract the lower eigenvalues, can be used unchanged with negative or zero masses. In the first method the iterated vector x is improved according to $Ly_1 = Mx_0$, $L^T x_1 = y_1$. In the latter we start by requiring to minimize $\mu^2 = (x^T Kx / x^T Mx)^2$ in order to obtain the lowest positive λ. The gradient of μ^2 is

$$\nabla \mu^2 = 4 \frac{x^T Kx}{(x^T Mx)^2} (Kx - \mu Mx). \tag{2}$$

Since $x^T Kx/(x^T Mx)^2$ is non negative we neglect it and use only the residual $r = Kx - \mu Mx$ as in [8, 9].

In direct time integration of the equations of motion solution by eigenvector expansion (mode superposition) is directly applicable with zero or negative masses. In difference methods, for stability reasons, one might have to use a lower integration rule [10] producing a positive diagonal M in order to arrive at a fully explicit difference scheme.

Otherwise, consider the heat conduction equation

$$KT + M\dot{T} = F(t) \tag{3}$$

to which the transformation $\theta = Te^{-\lambda_0 t}$ is applied. This leads to

$$(K + \lambda_0 M)\theta + M\dot{\theta} = e^{-\lambda_0 t}F(t) \tag{4}$$

where λ_0 can be adjusted such that the eigenvalues of $(K + \lambda_0 M)x = \mu Mx$ are positive. When an explicit scheme is used to solve (3) or (4), and that is a compelling reason for lumping M, stability is assured with $\delta t \leqslant 2/\mu_{\max}$.

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